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The Dirac spectrum of Bieberbach manifolds

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Abstract

The Dirac spectra and the eta invariants of three-dimensional Bieberbach manifolds are computed. Compact connected three-dimensional spin manifolds admitting parallel non-vanishing spinors are identified as flat tori. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Bieberbach manifolds are flat connected compact manifolds. In this article we study the spectrum of their Dirac operator.

At first, a review of Bieberbach's theorems is given. One of them states that every Bieberbach manifold M is covered by a flat torus T^n. We will see that spinors on M correspond to spinors on T^n satisfying a certain equivariance condition (2). The Dirac eigenvalues of M are contained in the Dirac spectrum of T^n, and in general the multiplicities of the eigenvalues of M are smaller than those of T^n. The Dirac spectrum of flat tori is well known, it depends on the choice of the spin structure. This result is due to Friedrich ([7], see also [1]). In order to calculate the eigenvalues on Bieberbach manifolds we lift the eigenspinors to the universal covering $\mathbb{R}^n$. By representation theory of finite groups we get formulae for the multiplicities of the Dirac eigenvalues of M. The method we use is related to the one Bär applied to compute the Dirac spectra of spherical space forms (see [2]).

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An explicit classification of three-dimensional orientable Bieberbach manifolds is available: there are only six distinct affine equivalence classes of such manifolds. For every case there exist several distinct spin structures which are classified in Theorem 3.3. In Theorems 5.4 and 5.7 we compute the Dirac spectra for all these cases. Eigenvalue 0 occurs only in the case of the flat torus T^3 with the trivial spin structure (see Theorem 5.1). Since the asymmetric components of these Dirac spectra have very simple forms it is easy to compute the eta invariants (Theorem 5.6).

An interesting observation can be made: there are examples of Bieberbach manifolds (G2,G4) for which a change of spin structures causes another qualitative behaviour of the Dirac spectrum. For some spin structures the spectrum is symmetric, for other spin structures it possesses an asymmetric component. This also illustrates the dependence of the eta invariants on the choice of the spin structure.

Section 6 is dedicated to parallel spinors. Two characterisations of flat tori are given: any three-dimensional compact connected spin manifold carrying a non-zero parallel spinor is a flat torus (Theorem 6.1). An n-dimensional oriented Bieberbach manifold for which the kernel of the Dirac operator has dimension $2^{[n/2]}$ is isometric to a torus (Theorem 6.2).

2. Flat manifolds

It is well known that any flat complete manifold M of dimension n is isometric to the quotient $G\backslash\mathbb{R}^n$, where G is a suitable subgroup of the Euclidean motions $E(n) := \mathrm{O}(n) \ltimes \mathbb{R}^n$.

For every element $g \in E(n)$ there exist $A \in \mathrm{O}(n)$ and $a \in \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ we have $gx = Ax + a$, and we write $g = (A, a)$.

One defines homomorphisms $r : E(n) \to \mathrm{O}(n)$ and $t : \mathbb{R}^n \to E(n)$ by $r(A, a) = A$ and $t(a) = (1, a)$. Obviously t is injective, therefore we may consider $\mathbb{R}^n$ as a subgroup of $E(n)$, the pure translations.

The subgroup $r(G) \subset O(n)$ is called the *holonomy* of G since it is isomorphic to the holonomy of M (see [5]).

A general result on the holonomy group of connected Riemannian manifolds states that a manifold is orientable if and only if its holonomy consists of isometries preserving the orientation of a given tangent space (see [10, p. 123]). So we get the following lemma.

Lemma 2.1. *A flat manifold $M = G\backslash\mathbb{R}^n$ is orientable iff $r(G) \subset \mathrm{SO}(n)$.*

Now we take a look at Bieberbach manifolds: a subgroup $G \subset E(n)$ acting properly discontinuously on $\mathbb{R}^n$ such that $G\backslash\mathbb{R}^n$ is compact is called a *Bieberbach group*. The structure of Bieberbach groups is described by the following theorem.

Theorem 2.2 (Bieberbach). *Let G be a Bieberbach group. Then the holonomy r(G) is finite and the set or pure translations in G defined as $\Gamma := G \cap \mathbb{R}^n$ is a lattice.*

From the proof given in [5, p. 17ff.] also two other things follow: the action of $r(G)$ on $\mathbb{R}^n$ leaves Γ invariant, i.e., $r(G)$ acts on Γ. Moreover, one has a short exact sequence $0 \to$

$\Gamma \to G \to r(G) \to 1$. Hence $\Gamma = \ker(r)$ is a normal subgroup of G with $r(G) \cong G/\Gamma$. This implies the following.

Theorem 2.3 (Bieberbach, [3]). *Every Bieberbach manifold is normally covered by a flat torus, and the covering map is a local isometry.*

The flat torus is $T^n := \Gamma\backslash\mathbb{R}^n$, and the action of $A \in r(G)$ on T^n is given as follows: choose $g \in G$ with $r(g) = A$ and set $A \cdot [x]_\Gamma := [gx]_\Gamma$. Thus we get $M^n \cong r(G)\backslash T^n$.

Bieberbach manifolds are well described by their fundamental groups as we see next.

Proposition 2.4. *Let $G_1, G_2 \subset E(n)$ be Bieberbach groups, let $\varphi : G_1 \to G_2$ be an isomorphism. Then there is an affine transformation $\alpha \in \mathrm{GL}(n) \ltimes \mathbb{R}^n$ such that for all $g \in G_1$: $\varphi(g) = \alpha g \alpha^{-1}$.*

Proof. See [5, p. 19]. □

We call two Bieberbach manifolds M_1 and M_2 *affine equivalent* if there exists a diffeomorphism $F : M_1 \to M_2$ whose lift to the universal Riemannian coverings $\pi_1 : \mathbb{R}^n \to M_1$, $\pi_2 : \mathbb{R}^n \to M_2$ is an affine linear map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\alpha} & \mathbb{R}^n \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{F} & M_2. \end{array}$$

A consequence of Proposition 2.4 is the following theorem.

Theorem 2.5 (Bieberbach). *Two Bieberbach manifolds are affine equivalent if their fundamental groups are isomorphic.*

The next theorem states that in principle one should be able to classify Bieberbach manifolds of a given dimension.

Theorem 2.6 (Bieberbach, [4]). *Let n be a positive integer. Then the number of affine equivalence classes of n-dimensional Bieberbach manifolds is finite.*

Proof. See [5, p. 65]. □

In the case of dimension $n \leq 3$ there are explicit classifications. Since we will do spin geometry we are interested in orientable Bieberbach manifolds only. In dimension 1 and 2 the only orientable Bieberbach manifolds are flat tori (see [12, p. 77]). In dimension 3 the classification is a bit more interesting.

Theorem 2.7 (Hantzsche, Wendt). *Let M be an orientable Bieberbach manifold of dimension three. Then M is affine equivalent to $G_i\backslash\mathbb{R}^3$ where G_i is one of the following six groups. In every case a basis of the lattice $\mathbb{R}^3 \cap G_i$ is denoted by $\{a_1, a_2, a_3\}$, the translation associated to a_j is called t_j, $j = 1, 2, 3$.*

	Generators of G_i	Defining relations
G1	t_1, t_2, t_3 with $\{a_1, a_2, a_3\}$ any basis of $\mathbb{R}^3$	$t_l t_k = t_k t_l \forall k, l$
G2	t_1, t_2, t_3, α with $a_1 \in [a_2, a_3]^\perp$, $\alpha = (A, \frac{1}{2}a_1)$, where $Aa_1 = a_1, Aa_2 = -a_2, Aa_3 = -a_3$	$t_l t_k = t_k t_l \forall k, l$, $\alpha^2 = t_1$, $\alpha t_2 \alpha^{-1} = t_2^{-1}$, $\alpha t_3 \alpha^{-1} = t_3^{-1}$
G3	t_1, t_2, t_3, α with $a_1 \in [a_2, a_3]^\perp$, $\|a_2\| = \|a_3\|$, a_2 and a_3 generate a plane regular hexagonal lattice, $\alpha = (A, \frac{1}{3}a_1)$, where $Aa_1 = a_1$, $Aa_2 = a_3$, $Aa_3 = -a_2 - a_3$	$t_l t_k = t_k t_l \forall k, l$, $\alpha^3 = t_1$, $\alpha t_2 \alpha^{-1} = t_3$, $\alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$
G4	t_1, t_2, t_3, α with a_1, a_2, a_3 mutually orthogonal, $\|a_2\| = \|a_3\|$, $\alpha = (A, \frac{1}{4}a_1)$, where $Aa_1 = a_1, Aa_2 = a_3, Aa_3 = -a_2$	$t_l t_k = t_k t_l \forall k, l$, $\alpha^4 = t_1$, $\alpha t_2 \alpha^{-1} = t_3$, $\alpha t_3 \alpha^{-1} = t_2^{-1}$
G5	t_1, t_2, t_3, α with $a_1 \in [a_2, a_3]^\perp$, $\|a_2\| = \|a_3\|$, a_2 and a_3 generate a plane regular hexagonal lattice, $\alpha = (A, \frac{1}{6}a_1)$, where $Aa_1 = a_1$, $Aa_2 = a_3$, $Aa_3 = -a_2 + a_3$	$t_l t_k = t_k t_l \forall k, l$, $\alpha^6 = t_1$, $\alpha t_2 \alpha^{-1} = t_3$, $\alpha t_3 \alpha^{-1} = t_2^{-1} t_3$
G6	$t_1, t_2, t_3, \alpha, \beta, \gamma$ with a_1, a_2, a_3 mutually orthogonal, $\alpha = (A, \frac{1}{2}a_1)$, $\beta = (B, \frac{1}{2}a_2 + \frac{1}{2}a_3)$, $\gamma = (C, \frac{1}{2}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3)$, where $Aa_1 = a_1, Aa_2 = -a_2, Aa_3 = -a_3$, $Ba_1 = -a_1, Ba_2 = a_2, Ba_3 = -a_3$, $Ca_1 = -a_1, Ca_2 = -a_2, Ca_3 = a_3$	$t_l t_k = t_k t_l \forall k, l$, $\alpha^2 = t_1$, $\alpha t_2 \alpha^{-1} = t_2^{-1}$, $\alpha t_3 \alpha^{-1} = t_3^{-1}$, $\beta t_1 \beta^{-1} = t_1^{-1}$, $\beta^2 = t_2$, $\beta t_3 \beta^{-1} = t_3^{-1}$, $\gamma t_1 \gamma^{-1} = t_1^{-1}$, $\gamma t_2 \gamma^{-1} = t_2^{-1}$, $\gamma^2 = t_3$, $\gamma\beta\alpha = t_1 t_3$

Proof. The generators and relations are given in [12, p. 117]. In [9] it is shown that these are the defining relations. □

The affine equivalence classes are denoted by G1, ..., G6, the associated Bieberbach groups are called $G_1, \ldots, G_6$. With some additional elementary considerations one gets the following theorem.

Theorem 2.8. *Every orientable Bieberbach manifold of dimension three is isometric to $G_i \backslash \mathbb{R}^3$, where G_i is one of the following groups, the parameters are to be chosen suitably.*

	Generators of G_i	*Basis of lattice*		*Parameters*
G1	t_1, t_2, t_3	a_1, a_2, a_3 *any basis of* $\mathbb{R}^3$		
G2	t_1, t_2, t_3, α *with* $\alpha = (A, \frac{1}{2}a_1)$	$a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, $a_3 = (T, S, 0)$	*A* π*-rotation about* z*-axis*	$H, L, S > 0, T \in \mathbb{R}$
G3	t_1, t_2, t_3, α *with* $\alpha = (A, \frac{1}{3}a_1)$	$a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, $a_3 = (-\frac{1}{2}L, (\sqrt{3}/2)L, 0)$	*A* $\frac{2\pi}{3}$*-rotation about* z*-axis*	$H, L > 0$
G4	t_1, t_2, t_3, α *with* $\alpha = (A, \frac{1}{4}a_1)$	$a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, $a_3 = (0, L, 0)$	*A* $\frac{\pi}{2}$*-rotation about* z*-axis*	$H, L > 0$
G5	t_1, t_2, t_3, α *with* $\alpha = (A, \frac{1}{6}a_1)$	$a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, $a_3 = (\frac{1}{2}L, (\sqrt{3}/2)L, 0)$	*A* $\frac{\pi}{3}$*-rotation about* z*-axis*	$H, L > 0$
G6	$t_1, t_2, t_3, \alpha, \beta, \gamma$ *with* $\alpha = (A, \frac{1}{2}a_1)$, $\beta = (B, \frac{1}{2}a_2 + \frac{1}{2}a_3)$, $\gamma = (C, \frac{1}{2}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3)$	$a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, $a_3 = (0, S, 0)$	*A* π*-rotation about* z*-axis, B* π*-rotation about* x*-axis, C* π*-rotation about* y*-axis*	$H, L, S > 0$

In particular the holonomy $r(G_i)$ *is cyclic for* $i = 2, \ldots, 5$.

3. Spin structures

Let $\mathbb{C}l(n)$ denote the Clifford algebra of $\mathbb{R}^n$, i.e., the complex algebra generated by $\mathbb{R}^n$ with the relations $v \cdot w + w \cdot v + 2\langle v, w\rangle = 0$ for all $v, w \in \mathbb{R}^n$. The space of an irreducible representation of $\mathbb{C}l(n)$ is $\Sigma_n = \mathbb{C}^K$ with $K = 2^{[n/2]}$. For $n = 3$ the representation can be given by the Pauli matrices (see [8]):

$$e_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

The group Spin(n) sits in $\mathbb{C}l(n)$:

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2k} \,|\, k \in \mathbb{N},\ \ |v_i| = 1 \ \text{ for all } i = 1, \ldots, 2k\},$$

and there is the double covering

$$\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n), \qquad u \mapsto (v \mapsto u \cdot v \cdot u^{-1}).$$

Next, we describe the spin structures on an oriented Bieberbach manifold $M = G\backslash\mathbb{R}^n$. We proceed as in [8]. Since $\mathbb{R}^n$ is simply connected it carries only one spin structure — the trivial one:

$$\begin{array}{ccc} P_{Spin}\mathbb{R}^n & = & \mathbb{R}^n \times Spin(n) \\ \downarrow \Lambda & & \downarrow id\times\lambda \\ P_{SO}\mathbb{R}^n & = & \mathbb{R}^n \times SO(n) \end{array}$$

where $P_{\mathrm{SO}}\mathbb{R}^n$ denotes the set of all oriented orthonormal bases of tangent spaces of $\mathbb{R}^n$. The action of G on $P_{\mathrm{SO}}\mathbb{R}^n$ is given by

$$g(x, (v_1, \dots v_n)) = (gx, (\mathrm{d}g(v_1), \dots , \mathrm{d}g(v_n))) = (gx, r(g)(v_1, \dots , v_n))$$

for all $x \in \mathbb{R}^n$, $(v_1, \dots , v_n) \in \mathrm{SO}(n)$. We get $P_{\mathrm{SO}}M \cong G\backslash P_{\mathrm{SO}}\mathbb{R}^n$. Now there are two lifts $g^{\pm}$ of g such that

$$\begin{array}{ccc} P_{Spin}\mathbb{R}^n & \xrightarrow{g^{\pm}} & P_{Spin}\mathbb{R}^n \\ \downarrow \Lambda & & \downarrow \Lambda \\ P_{SO}\mathbb{R}^n & \xrightarrow{g} & P_{SO}\mathbb{R}^n \end{array}$$

Proposition 3.1. *There is a* one-to-one *correspondence between the spin structures on M and the actions* α *of G on* $P_{\mathrm{Spin}}\mathbb{R}^n$ *with*: $\alpha(g) \in \{g^{\pm}\}$ *for all* $g \in G$.

Proof. See [8, p. 46]. □

The spin structure associated to such an α is given by

$$G\backslash P_{\mathrm{Spin}}\mathbb{R}^n \to \quad G\backslash P_{\mathrm{SO}}\mathbb{R}^n \cong P_{\mathrm{SO}}M.$$

For $g^{\pm}$ we can find $A^{\pm} \in \lambda^{-1}(r^{-1}(g))$ such that for all $(x, s) \in \mathbb{R}^n \times \mathrm{Spin}(n)$,

$$g^{\pm} = (gx, A^{\pm}s).$$

From Proposition 3.1 one gets the following proposition.

Proposition 3.2. *The spin structures on* $M = G\backslash\mathbb{R}^n$ *with the induced orientation are in bijective relation to the homomorphisms* $\varepsilon : G \to \mathrm{Spin}(n)$ *with*

$$\begin{array}{ccc} & & Spin(n) \\ & \overset{\varepsilon}{\nearrow} & \downarrow \lambda \\ G & \xrightarrow{r} & SO(n)\,. \end{array}$$

Given a homomorphism ε with $r = \lambda \circ \varepsilon$ one defines an action α on $\mathbb{R}^n \times \mathrm{Spin}(n)$ via $\alpha(g)(x, s) = (gx, \varepsilon(g)s)$, and one gets a spin structure as described above.

In order to classify the spin structures on oriented Bieberbach manifolds of dimension 3 we have to recall a simple fact concerning groups: let a group G be given by generators and relations, let ε be a map from the set of the generators of G into a group H. Then ε extends to a homomorphism $G \to H$ if and only if the same relations hold for the ε-images of the generators. Considering the λ-preimages of $r(g)$ for every generator g of G and checking the relations we get the following theorem.

Theorem 3.3. *Let $G_i \subset \mathrm{SO}(3) \ltimes \mathbb{R}^n$ be a Bieberbach group as in Theorem* 2.8. *Then one gets every spin structure on $M = G_i\backslash\mathbb{R}^3$ by taking one of the homomorphisms $\varepsilon : G_i \to$* Spin(3) *with $r = \lambda \circ \varepsilon$ whose values on the generators of G_i are given by the following*:

G1	$a_1 \mapsto \delta_1,\ a_2 \mapsto \delta_2,\ a_3 \mapsto \delta_3$	$\delta_1, \delta_2, \delta_3 \in \{\pm 1\}$
G2	$a_1 \mapsto -1,\ a_2 \mapsto \delta_2,\ a_3 \mapsto \delta_3,\ \alpha \mapsto \delta_1 e_1 e_2$	$\delta_1, \delta_2, \delta_3 \in \{\pm 1\}$
G3	$a_1 \mapsto -\delta_1,\ a_2 \mapsto 1,\ a_3 \mapsto 1,\ \alpha \mapsto \delta_1(\frac{1}{2} + (\sqrt{3}/2)e_1 e_2)$	$\delta_1 \in \{\pm 1\}$
G4	$a_1 \mapsto -1,\ a_2 \mapsto \delta_2,\ a_3 \mapsto \delta_2,\ \alpha \mapsto \delta_1(\sqrt{2}/2 + (\sqrt{2}/2)e_1 e_2)$	$\delta_1, \delta_2 \in \{\pm 1\}$
G5	$a_1 \mapsto -1,\ a_2 \mapsto 1,\ a_3 \mapsto 1,\ \alpha \mapsto \delta_1(\sqrt{3}/2 + \frac{1}{2} e_1 e_2)$	$\delta_1 \in \{\pm 1\}$
G6	$a_1 \mapsto -1, a_2 \mapsto -1, a_3 \mapsto -1,\ \alpha \mapsto \delta_1 e_1 e_2, \beta \mapsto \delta_2 e_2 e_3, \gamma \mapsto \delta_3 e_3 e_1$	$\delta_1, \delta_2, \delta_3 \in \{\pm 1\}$ *with* $\delta_1 \cdot \delta_2 \cdot \delta_3 = 1$

In particular, in the cases G1 *and* G2 *there are eight distinct spin structures, for* G3 *and* G5 *there are two, and for* G4 *and* G6 *there are four*.

In the cases G2–G5 one can write $\varepsilon(\alpha)$ alternatively as $\varepsilon(\alpha) = \delta_1(\cos(\varphi/2) + \sin(\varphi/2) e_1 e_2)$ with $\varphi = 2\pi/k$ and $k = \#r(G_i)$.

4. Spectra

Now, let $M = G\backslash\mathbb{R}^n$ be a Bieberbach manifold with the spin structure given by $\varepsilon : G \to \mathrm{Spin}(n)$. The spinor bundle of M is the associated bundle $\Sigma M := P_{\mathrm{Spin}} M \times_{\mathrm{Spin}(n)} \Sigma_n$. For $\mathbb{R}^n$ it is trivial: $\Sigma\mathbb{R}^n \cong \mathbb{R}^n \times \Sigma_n$.

We may identify $\Sigma M = G\backslash\Sigma\mathbb{R}^n$, where $g \in G$ acts on $\Sigma\mathbb{R}^n$ by $g(x,\sigma) = (gx, \varepsilon(g)\sigma)$ for all $(x,\sigma) \in \Sigma\mathbb{R}^n$. Therefore, one can consider spinors on M as maps $\Psi : \mathbb{R}^n \to \Sigma_n$ satisfying for all $g \in G$:

$$\Psi = \varepsilon(g)\Psi \circ g^{-1}. \tag{2}$$

Let ∇ denote the Levi-Civita connection for spinors, and let D be the Dirac operator on M.

For $T^n = \Gamma\backslash\mathbb{R}^n$ the spectrum of D^2 is already known (see [7]): let the spin structure of T^n be given by $\varepsilon : \Gamma \to \{\pm 1\} \subset \mathrm{Spin}(n)$, let $a_1^*, \dots, a_n^*$ be a basis of the dual lattice Γ^* of Γ. We define

$$a_\varepsilon := \frac{1}{2} \sum_{l \text{ with } \varepsilon(a_l)=-1} a_l^*. \tag{3}$$

The D^2-eigenspinors on T^n are given by

$$\Psi_b^j : \mathbb{R}^n \to \Sigma_n, \qquad x \mapsto \exp(2\pi\mathrm{i}\langle b, x\rangle)\sigma^j,$$

where $b \in \Gamma^* + a_\varepsilon$, and $\{\sigma^j \mid j = 1, \dots, 2^{[n/2]}\}$ is the standard basis of Σ_n. We denote the corresponding D^2-eigenspace $E_b(D^2) := \mathrm{span}\{\Psi_b^j\}_j$. For $\Phi \in E_b(D^2)$, $b \neq 0$, the Dirac operator is given by the Clifford multiplication with $2\pi\mathrm{i}b$:

$$D\Phi = 2\pi\mathrm{i}b \cdot \Phi.$$

Let $E_{b\pm}(D)$ be the set of all $\Psi \in E_b(D^2)$ with $D\Psi = \pm 2\pi|b|\Psi$. Clearly, we get $E_b(D^2) = E_{b+}(D) \oplus E_{b-}(D)$, i.e., a decomposition into eigenspaces of D. It is known that the Dirac spectrum of T^n is symmetric for every possible spin structure (see [1]). Analogously as in [2] we define projection operators $F^\pm : E_b(D^2) \to E_{b\pm}(D)$ by

$$F^\pm\Psi := \left(1 \pm \frac{1}{2\pi|b|}D\right)\Psi = \left(1 \pm \mathrm{i}\frac{b}{|b|}\right)\Psi.$$

Since $F^\pm$ is surjective we obtain generators of $E_{b\pm}(D)$

$$\Phi_{b\pm}^j := F^\pm\Psi_b^j = \left(1 \pm \mathrm{i}\frac{b}{|b|}\right)\Psi_b^j, \quad j = 1, \dots, 2^{[n/2]}.$$

In the case $b = 0 \in \Gamma^* + a_\varepsilon$ one gets $E_0(D^2) = E_0(D)$, and generators of $E_0(D)$ are given by $\Phi_0^j := \sigma^j,\ j = 1, \dots, 2^{[n/2]}$.

For $b \neq 0$ one obtains an isomorphism $E_{b+}(D) \cong E_{b-}(D)$ by the following: choose $c \in \mathbb{R}^n$ perpendicular to b with $|c| = 1$. Let $M_c : E_b(D^2) \to E_b(D^2)$ denote the Clifford multiplication with c. Then, M_c and D anticommute:

$$M_c D\Phi = -DM_c\Phi \quad \text{for all } \Phi \in E_b(D^2).$$

Therefore, $M_c : E_{b\pm}(D) \to E_{b\mp}(D)$. Now $(M_c)^2 = \mathrm{id}$ implies that M_c is an isomorphism. Consequently, $\dim(E_{b\pm}(D)) = \frac{1}{2}2^{[n/2]}$.

Next, we consider $M = G\backslash\mathbb{R}^n$ with spin structure given by $\varepsilon : G \to \mathrm{Spin}(n)$, the general case. Theorem 2.3 tells us that M is covered by the flat torus $T^n = \Gamma\backslash\mathbb{R}^n$ with $\Gamma = \mathbb{R}^n \cap G$. The spin structure on T^n is given by $\varepsilon|_\Gamma : \Gamma \to \mathrm{Spin}(n)$. The spinors on T^n satisfy the equivariance condition (2) only for $g \in \Gamma$, in general they are not equivariant for all $g \in G$. To find the G-equivariant spinors we define an action of $r(G)$ on the spinors on T^n. For $A \in r(G)$ we choose $g \in G$ with $r(g) = A$, and for a spinor Ψ on T^n we set

$$A\Psi := \varepsilon(g)\Psi \circ g^{-1}.$$

One can show that by this one gets a welldefined action on the space of spinors on T^n. Obviously, the $r(G)$-equivariant spinors on T^n correspond to the spinors on M.

Let $a_1, \dots, a_n$ be a basis of Γ and $a_1^*, \dots, a_n^*$ be the dual basis. For the sake of simplicity we write a_ε instead of $a_{\varepsilon|_\Gamma}$. Now, we calculate $A\Phi^j_{b\pm}$ for $b \in \Gamma^* + a_\varepsilon,\ b \neq 0,\ j = 1, \dots, 2^{[n/2]}$. For a chosen $g \in G$ with $r(g) = A$, i.e., $g = (A, a)$, we get the following lemma.

Lemma 4.1. $A\Phi^j_{b\pm} = \exp(2\pi \mathrm{i}\langle Ab, a\rangle)\left(1 \pm \frac{Ab}{|Ab|}\right)\left(\varepsilon(g)\Psi^j_{Ab}\right).$

Before we prove this lemma, it should be noted that the invariance of $\Gamma \subset \mathbb{R}^n$ under $r(G) \subset \mathrm{SO}(n)$ implies that $\Gamma^* \subset \mathbb{R}^n$ is invariant under $r(G)^* = r(G)$. From the fact that $A\Phi^j_{b\pm}$ is a spinor on T^n it follows that $Ab \in \Gamma^* + a_\varepsilon$.

Proof. First we get for all $x \in \mathbb{R}^n$:

$$\left(\Psi^j_b \circ g^{-1}\right)(x) = \exp(2\pi \mathrm{i}\langle b, A^{-1}(x-a)\rangle)\sigma^j = \exp(-2\pi \mathrm{i}\langle Ab, a\rangle)\Psi^j_{Ab}(x).$$

Next, we only use the definitions

$$\begin{aligned} A\Phi^j_{b\pm} &= \varepsilon(g)\Phi^j_{b\pm} \circ g^{-1} = \varepsilon(g)\left(1 \pm \mathrm{i}\frac{b}{|b|}\right)\Psi^j_b \circ g^{-1} \\ &= \left(1 \pm \mathrm{i}\frac{1}{|b|}\varepsilon(g)b\varepsilon(g)^{-1}\right)\varepsilon(g)\Psi^j_b \circ g^{-1}. \end{aligned}$$

From $r = \lambda \circ \varepsilon$ it follows that $\varepsilon(g)b\varepsilon(g)^{-1} = (\lambda \circ \varepsilon(g))(b) = r(g)b = Ab$. Furthermore, $A \in \mathrm{SO}(n)$ implies that $|b| = |Ab|$. Finally we get

$$A\Phi^j_{b\pm} = \left(1 \pm \frac{Ab}{|Ab|}\right)\left(\varepsilon(g)\exp(-2\pi \mathrm{i}\langle Ab, a\rangle)\Psi^j_{Ab}\right).$$

□

We can write $A\Phi^j_{b\pm} = F^\pm \exp(-2\pi \mathrm{i}\langle Ab, a\rangle)\varepsilon(g)\Psi^j_{Ab})$. Hence for all $\Phi \in E_{b\pm}(D)$ we have $A\Phi \in E_{Ab\pm}(D)$. The following theorem is useful to compute the symmetric component of the Dirac spectrum of Bieberbach manifolds.

Theorem 4.2. *Suppose that for* $b \in \Gamma^* + a_\varepsilon, b \neq 0$, *one has* $\#r(G) = \#r(G)b$, i.e., $r(G)$ *acts on the* $r(G)$*-orbit of* b *without fixed points. Consider*

$$V := \bigoplus_{A \in r(G)} E_{Ab}(D^2).$$

Then, the dimensions of the subspaces of V consisting of D-eigenspinors of M associated to the eigenvalues $\pm 2\pi|b|$ *are given by*

$$\mathrm{mult}(\pm 2\pi|b|, D|_V) = \frac{1}{2}2^{[n/2]}.$$

Proof. Theorem 2.2 states that $r(G)$ is finite: $r(G) = \{A_1, \ldots, A_k\}$ with $k = \#r(G)$. As by assumption the points $A_1 b, \ldots, A_k b$ are pairwise distinct, the spaces $E_{A_j b}(D^2)$ are mutually orthogonal. Therefore, V is a direct sum. We define

$$V^{\pm} := \bigoplus_{A\in r(G)} E_{Ab\pm}(D).$$

The action of $r(G)$ induces representations $\rho^{\pm} : r(G) \to \mathrm{GL}(V^{\pm})$. Let $\chi^{\pm}$ denote the associated characters. From Lemma 4.1 it follows that $\chi^{\pm}(A) = \mathrm{tr}(\rho^{\pm}(A)) = 0$ for $A \in r(G)$, $A \neq \mathrm{id}$. The subspace of D-eigenspinors is the space on which $r(G)$ acts trivially. Hence,

$$\begin{aligned}\mathrm{mult}(\pm 2\pi|b|, D|_V) &= \langle\chi^{\pm}, 1\rangle = \frac{1}{\#r(G)}\sum_{A\in r(G)}\chi^{\pm}(A) = \frac{1}{k}\chi^{\pm}(\mathrm{id}) = \frac{1}{k}\dim(V^{\pm})\\ &= \frac{1}{k}\cdot\frac{1}{2}\cdot k\cdot 2^{[n/2]}.\end{aligned}$$

□

Corollary 4.3. *Assume the action of r(G) on* $\Gamma^* + a_\varepsilon$ *is free, then the spectrum of the Dirac operator on M is symmetric.*

In the case of $b = 0 \in \Gamma^* + a_\varepsilon$ the action of $A \in r(G)$ is given by

$$A\Psi = \varepsilon(g)\Psi \in E_0(D)$$

for every $\Psi \in E_0(D) = \Sigma_n$ and $g \in r^{-1}(A) \subset G$. The kernel of the Dirac operator on M is the subspace of $r(G)$-invariant spinors in $E_0(D)$, its dimension is

$$\dim(\ker(D)) = \frac{1}{\#r(G)}\sum_{A\in r(G)}\chi(A),$$

where χ denotes the character of the representation $r(G) \to \mathrm{GL}(E_0(D))$.

5. Spectra in dimension 3

In the following we will use the preceding preparations to compute the Dirac spectrum of three-dimensional Bieberbach manifolds.

For a_1, a_2, a_3 given in Theorem 2.8 we get the dual basis a_1^*, a_2^*, a_3^*:

G2	$a_1^* = (0, 0, 1/H)$, $a_2^* = (1/L, -T/SL, 0)$, $a_3^* = (0, 1/S, 0)$
G3	$a_1^* = (0, 0, 1/H)$, $a_2^* = (1/L, (1/3)\sqrt{3}(1/L), 0)$, $a_3^* = (0, (2/3)\sqrt{3}(1/L), 0)$
G4	$a_1^* = (0, 0, 1/H)$, $a_2^* = (1/L, 0, 0)$, $a_3^* = (0, 1/L, 0)$
G6	$a_1^* = (0, 0, 1/H)$, $a_2^* = (1/L, 0, 0)$, $a_3^* = (0, 1/S, 0)$

We obtain distinct a_ε for the distinct spin structures given by $\delta_i \in \{\pm 1\}$ as in Theorem 3.3:

	Spin structures	a_ε
G2	$\delta_1 \in \{\pm 1\},\ \ \delta_2 = 1,\ \ \delta_3 = 1$	$\frac{1}{2}a_1^* = (0, 0, 1/2H)$
	$\delta_1 \in \{\pm 1\},\ \ \delta_2 = -1,\ \ \delta_3 = 1$	$\frac{1}{2}a_1^* + \frac{1}{2}a_2^*$ $= (1/2L, -T/2SL, 1/2H)$
	$\delta_1 \in \{\pm 1\},\ \ \delta_2 = 1,\ \ \delta_3 = -1$	$\frac{1}{2}a_1^* + \frac{1}{2}a_3^* = (0, 1/2S, 1/2H)$
	$\delta_1 \in \{\pm 1\},\ \ \delta_2 = -1,\ \ \delta_3 = -1$	$\frac{1}{2}a_1^* + \frac{1}{2}a_2^* + \frac{1}{2}a_3^*$ $= (1/2L, 1/2S - T/2SL, 1/2H)$
G3	$\delta_1 = 1$	$\frac{1}{2}a_1^* = (0, 0, 1/2H)$
	$\delta_1 = -1$	$0 = (0, 0, 0)$
G4	$\delta_1 \in \{\pm 1\},\ \ \delta_2 = 1$	$\frac{1}{2}a_1^* = (0, 0, 1/2H)$
	$\delta_1 \in \{\pm 1\},\ \ \delta_2 = -1$	$\frac{1}{2}a_1^* + \frac{1}{2}a_2^* + \frac{1}{2}a_3^* = (1/2L, 1/2L, 1/2H)$
G5	$\delta_1 \in \{\pm 1\}$	$\frac{1}{2}a_1^* = (0, 0, 1/2H)$
G6	$\delta_1, \delta_2, \delta_3 \in \{\pm 1\}$ with $\delta_1\delta_2\delta_3 = 1$	$\frac{1}{2}a_1^* + \frac{1}{2}a_2^* + \frac{1}{2}a_3^* = (1/2L, 1/2S, 1/2H)$

We consider the case G6: for $b \in \Gamma^* + a_\varepsilon$ one has $\#r(G)b = 4 = \#r(G)$. Therefore, $r(G)$ acts on $\Gamma^* + a_\varepsilon$ without fixed points. We apply Corollary 4.3 and note that in this case the spectrum is symmetric.

The computation of the Dirac spectra is done in three steps.

First, we investigate when the kernel of D is non-trivial. Then we observe in which cases $\Gamma^* + a_\varepsilon$ possesses some non-maximal $r(G)$-orbits, i.e., orbits $r(G)b$ with $\#r(G)b < \#r(G)$. Theorem 4.2 tells us that only these orbits can have a contribution to the asymmetric component of the spectrum of D. At last, we just have to count the maximal orbits in $\Gamma + a_\varepsilon$ to get the symmetric component.

To determine the kernel of D we only have to observe the cases with $0 \in \Gamma^* + a_\varepsilon$: these are the flat torus with the trivial spin structure and G3 with the spin structure given by $\delta_1 = -1$. In the second case the holonomy is $r(G) = \{1, A, A^2\}$, where A is the $(2\pi/3)$-rotation around the z-axis. As an r-preimage of A we choose $\alpha = (A, \frac{1}{3}a_1)$ (compare Theorem 2.8). Then by Theorem 3.3, $\varepsilon(\alpha) = \frac{1}{2}(1 + \sqrt{3}e_1e_2)$. Using the representation defined by (1) we get

$$\rho(A) = -\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \quad \text{and} \quad \rho(A^2) = \rho(A)^2 = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

The associated character is given by

$$\chi(1) = 2, \quad \chi(A) = -1 \quad \text{and} \quad \chi(A^2) = -1.$$

Hence, $\dim(\ker(D)) = \frac{1}{3}(2 - 1 - 1) = 0$, and we have shown the following theorem.

Theorem 5.1. *The only Bieberbach manifolds of dimension 3 which are spin and on which D has a non-trivial kernel for a suitable choice of the spin structure are flat tori.*

Next, we will compute the asymmetric component of the Dirac spectrum. As for G6 the spectrum of D is symmetric it suffices to study the cases G2–G5 which are very similar: $r(G)$ is cyclic and consists of rotations around the z-axis. Consequently, an orbit $r(G)b$ is maximal if and only if b sits on the z-axis which means b is of the form $b = \beta e_3$, $\beta \in \mathbb{R}$. For $\Gamma^* + a_\varepsilon$ possessing points on the z-axis the only possibilities are $a_\varepsilon = 0$ or $a_\varepsilon = \frac{1}{2}a_1^*$. We get the following lemma.

Lemma 5.2. *Asymmetric D-spectra are only possible in the following eight cases*:

G2	$\delta_1 \in \{\pm 1\}, \quad \delta_2 = 1, \quad \delta_3 = 1$
G3	$\delta_1 \in \{\pm 1\}$
G4	$\delta_1 \in \{\pm 1\}, \quad \delta_2 = 1$
G5	$\delta_1 \in \{\pm 1\}$

Next, we will only consider these eight cases. For $b \in \Gamma^* + a_\varepsilon$ sitting on the z-axis, $b \neq 0$, one has $Ab = b$ for all $A \in r(G)$. Hence, $E_{b\pm}(D) = E_{Ab\pm}(D)$, and by Lemma 4.1 one gets representations $\rho^\pm : r(G) \to \mathrm{GL}(E_{b\pm}(D))$ with characters $\chi^\pm$. As $\dim_{\mathbb{C}} E_{b\pm}(D) = \frac{1}{2}2^{[3/2]} = 1$, we have representations of a cyclic group on a one-dimensional linear space. Let the order of $r(G)$ be denoted by $k = \#r(G)$, let A be a generator of $r(G)$ as in Theorem 2.8. The dimension of the subspace of $r(G)$-equivariant spinors in $E_{b\pm}(D)$ is

$$\langle \chi^\pm, 1 \rangle = \frac{1}{k}\sum_{l=0}^{k-1} \chi^\pm(A^l) = \frac{1}{k}\sum_{l=0}^{k-1} (\chi^\pm(A))^l. \tag{4}$$

We write $b = \beta e_3$ with $b \in \mathbb{R}\backslash\{0\}$ and get a basis of $E_{b\pm}(D)$:

$$\Phi_{b\pm}^1 = \left(1 \pm \mathrm{i}\frac{b}{|b|}\right)\Psi_b^1 = f_b(1 \pm \mathrm{i} \cdot \mathrm{sgn}(\beta) e_3)\sigma^1,$$

where f_b denotes the map $\mathbb{R}^3 \to \mathbb{C}$, $x \mapsto \exp(2\pi \mathrm{i}\langle x, b\rangle)$. Using (1) we get

$$\Phi_{b\pm}^1 = f_b(\sigma^1 \mp \mathrm{i} \cdot \mathrm{sgn}(\beta)\sigma^2) \neq 0.$$

Just like in Theorem 2.8 we take $\alpha = (A, (1/k)a_1)$ as an r-preimage of A. By Lemma 4.1 it follows that

$$A\Phi_{b\pm}^1 = \exp(-2\pi \mathrm{i}(1/k)\langle b, a\rangle)\left(1 \pm \mathrm{i}\frac{b}{|b|}\right)\varepsilon(a)\Psi_b^1. \tag{5}$$

For the representation given in (1) the actions of $e_1 \cdot e_2$ and $-e_3$ on Σ_3 are the same. Using Theorem 3.3 and setting $\varphi := 2\pi/k$ we obtain

$$\begin{aligned}\left(1\pm \mathrm{i}\frac{b}{|b|}\right)\varepsilon(a)\Psi_b^1 &= (1\pm \mathrm{i}\cdot \mathrm{sgn}(\beta)e_3)\delta_1\left(\cos\frac{\varphi}{2}+\sin\frac{\varphi}{2}e_1e_2\right)\Psi_b^1\\ &= (1\pm \mathrm{i}\cdot \mathrm{sgn}(\beta)e_3)\delta_1\left(\cos\frac{\varphi}{2}-\sin\frac{\varphi}{2}e_3\right)\Psi_b^1\\ &= \delta_1\left(\cos\frac{\varphi}{2}\pm \mathrm{i}\cdot \mathrm{sgn}(\beta)\sin\frac{\varphi}{2}\right)(1\pm \mathrm{i}\cdot \mathrm{sgn}(\beta)e_3)\Psi_b^1\\ &= \delta_1\exp(\pm \mathrm{i}(\varphi/2)\,\mathrm{sgn}(\beta))\left(1\pm \mathrm{i}\frac{b}{|b|}\right)\Psi_b^1\\ &= \delta_1\exp(\pm \mathrm{i}(\varphi/2)\,\mathrm{sgn}(\beta))\Phi_{b\pm}^1.\end{aligned}$$

Plugging this into (5) one gets $A\Phi^1_{b\pm} = \delta_1 \exp(-2\pi \mathrm{i}(1/k)\langle b, a_1\rangle)\cdot \exp(2\pi \mathrm{i}(1/2k)(\pm\mathrm{sgn}(\beta)))\Phi^1_{b\pm}$. In each case of Lemma 5.2 we can find $H > 0$ with $e_3 = Ha_1^*$, and thus $b = (\beta H)a_1^*$. Hence the character of A is

$$\chi^{\pm}(A) = \delta_1 \exp\left(2\pi \mathrm{i}\frac{1}{k}\left(-\beta H \pm \frac{1}{2}\mathrm{sgn}(\beta H)\right)\right). \tag{6}$$

The next lemma is a direct consequence of the geometric summation, and it will be useful in the following computations.

Lemma 5.3. *Let $\xi\in\mathbb{C}$ be a kth root of 1, $\xi^k = 1$, then*

$$\frac{1}{k}\sum_{l=0}^{k-1}\xi^l = \begin{cases}1 & \textit{if } \xi = 1,\\ 0 & \textit{otherwise}.\end{cases}$$

Theorem 5.4. *Only in the eight cases of Lemma 5.2 the spectrum of D has an asymmetric component $\mathcal{B}$.* Let *$k = \#r(G)$ denote the order of the holonomy. Then one gets for* G2,G3,G4,G5 *with the spin structure given by $\delta_1 = 1$:*

$$\mathcal{B} = \left\{2\pi\frac{1}{H}\left(k\mu+\frac{1}{2}\right)|\mu\in\mathbb{Z}\right\}$$

for all $\mu\in\mathbb{Z}$ the multiplicities are

$$\mathrm{mult}\left(2\pi\frac{1}{H}\left(k\mu+\frac{1}{2}\right), D\right) = 2.$$

If one chooses the spin structure given by $\delta_1 = -1$, one obtains

$$\mathcal{B} = \left\{2\pi\frac{1}{H}\left(k\mu+\frac{k+1}{2}\right)|\mu\in\mathbb{Z}\right\},$$

and for $\mu\in\mathbb{Z}$ the multiplicity is

$$\mathrm{mult}\left(2\pi\frac{1}{H}\left(k\mu+\frac{k+1}{2}\right), D\right) = 2$$

Proof. We only have to plug (6) into (4) and consider the distinct cases. We note that in all cases except G3 with $\delta_1 = -1$ one gets $b = (z + \frac{1}{2})a_1^*$ with $z \in \mathbb{Z}$. For G3 with $\delta_1 = -1$ one can write $b = za_1^*$, where $z \in \mathbb{Z}$, $z \neq 0$.

1. $\delta_1 = 1$. For $b = (z + \frac{1}{2})a_1^*$, i.e. $(\beta H) = z + \frac{1}{2}$ it follows from (6):

$$\chi^{\pm}(A) = \exp\left(2\pi\mathrm{i}\frac{1}{k}\left(-z - \frac{1}{2} \pm \frac{1}{2}\mathrm{sgn}\left(z + \frac{1}{2}\right)\right)\right).$$

We put

$$\nu_z^{\pm} := \mathrm{mult}\left(\pm 2\pi\left|\left(z + \frac{1}{2}\right)a_1^*\right|,\ D|_{V_{z\pm}}\right), \quad \text{where } V_{z\pm} := E_{((z+1/2)a_1^*)\pm}(D).$$

Together with (4) Lemma 5.3 yields:

$$\nu_z^{\pm} = \begin{cases} 1 & \text{if } \chi^{\pm}(A) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi^{\pm}(A) = 1$ is equivalent to $-z - \frac{1}{2} \pm \frac{1}{2}\mathrm{sgn}(z + \frac{1}{2}) \in k\mathbb{Z}$, we get for $z \geq 0$:

$$\nu_z^{+} = \begin{cases} 1 & \text{if } z \equiv 0 \mod k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_z^{-} = \begin{cases} 1 & \text{if } z \equiv -1 \mod k, \\ 0 & \text{otherwise,} \end{cases}$$

and for $z < 0$: $\quad \nu_z^{+} = \begin{cases} 1 & \text{if } z \equiv -1 \mod k, \\ 0 & \text{otherwise,} \end{cases}$

$$\nu_z^{-} = \begin{cases} 1 & \text{if } z \equiv 0 \mod k, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, only $z = \mu k$ and $z = \mu k - 1$, $\mu \in \mathbb{Z}$, make a contribution to the spectrum. One gets the positive eigenvalues exactly from those z with $z = \mu k$ and $z = -\mu k - 1$, $\mu \geq 0$, and the negative ones exactly from $z = \mu k$ and $z = -\mu k - 1$ for $\mu < 0$. As $|a_1^*| = 1/H$, the eigenvalues are $2\pi(1/H)(\mu k + \frac{1}{2})$, $\mu \in \mathbb{Z}$. For $\mu \geq 0$ the multiplicities are:

$$\mathrm{mult}\left(2\pi\frac{1}{H}\left(k\mu + \frac{1}{2}\right), D\right) = \nu_{z_1}^{+} + \nu_{z_2}^{+} = 1 + 1 = 2,$$

where $z_1 = k\mu$ and $z_2 = -k\mu - 1$. In the same way one obtains the multiplicities 2 for $\mu < 0$.

2. $\delta_1 = -1$. As $\delta_1 = \exp(2\pi\mathrm{i}\frac{1}{2})$, the character is given by

$$\chi^{\pm}(A) = \exp\left(2\pi\mathrm{i}\frac{1}{k}\left(-(\beta H) \pm \frac{1}{2}\mathrm{sgn}(\beta H) + \frac{k}{2}\right)\right).$$

Hence, $\chi^{\pm}(A) = 1 \Leftrightarrow -(\beta H) \pm \frac{1}{2}\mathrm{sgn}(\beta H) + \frac{1}{2}k \equiv 0 \mod k$, then the following computations are analogous as above. One has to observe that for G2,G4,G5 one has $(\beta H) \in \mathbb{Z} + \frac{1}{2}$ and $\frac{1}{2}k \in \mathbb{Z}$, and for G3: $(\beta H) \in \mathbb{Z}$ and $\frac{1}{2}k = 1 + \frac{1}{2}$. □

Now, the eta invariants are easily computed. It is clear that for symmetric spectra the eta invariants vanish.

Lemma 5.5. *Assume the spectrum has an asymmetric component of the form* $\mathcal{B} = \{r(\mu + \alpha) | \mu \in \mathbb{Z}\}$ *with* $\alpha \in (0, 1)$ *and* $r > 0$ *such that each eigenvalue in* $\mathcal{B}$ *has the same multiplicity A. Then the eta invariant is* $\eta = A(1 - 2\alpha)$.

Proof. For $\mathrm{Re}(z) \gg 0$ one gets for the eta function:

$$\eta(z) = \sum_{\substack{\lambda \in \mathrm{spec}(D) \\ \lambda \neq 0}} \mathrm{sgn}(\lambda) \frac{\mathrm{mult}(\lambda, D)}{|\lambda|^z}$$

$$= \sum_{\lambda \in \mathcal{B}} \mathrm{sgn}(\lambda) \frac{A}{|\lambda|^z} = A \frac{1}{r^z} \left(\sum_{k=0}^{\infty} \frac{1}{(k + \alpha)^z} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \alpha)^z} \right).$$

These two series are known as generalized zeta functions (see [11, p. 265ff.]). They have meromorphic extensions on $\mathbb{C}$ without poles in $z = 0$. Let $\zeta(z, a)$ denote the function defined by $\sum_{k=0}^{\infty} 1/(k + \alpha)^z$ for $\mathrm{Re}(z) \gg 0$. One gets for the extension: $\zeta(0, a) = \frac{1}{2} - \alpha$.

Hence, the eta invariant is $\eta(0) = A\left(\frac{1}{2} - \alpha - \frac{1}{2} + (1 - \alpha)\right)$. □

Theorem 5.4 tells us that only in the case of Lemma 5.2 an asymmetric component $\mathcal{B}$ occurs, $\mathcal{B}$ has the form as in Lemma 5.5 if one takes $r = 2\pi(k/H)$ and $\alpha = 1/2k$ for $\delta_1 = 1$, and $r = 2\pi(k/H)$ and $\alpha = (k + 1)/2k$ in the case $\delta_1 = -1$. This yields the following theorem.

Theorem 5.6. *The eta invariant of a three-dimensional oriented Bieberbach manifold is zero except in the eight cases of Lemma* 5.2: *for* G2–G5 *with the spin structure given by* $\delta_1 = 1$ *the eta invariant is* $\eta = 2(1 - 1/k) = 2 - 2/k$, *and for* $\delta_1 = -1$ *it is* $\eta = 2(1 - (k + 1)/k) = -2/k$.

It remains to determine the symmetric components of the spectra. So far, we have just considered the points in $\Gamma^* + a_\varepsilon$ sitting on the z-axis. All the other points belong to maximal orbits. By Theorem 4.2 every maximal orbit $r(G)b$ contributes the eigenvalues $2\pi|b|$ and $-2\pi|b|$, with multiplicity $1 = \frac{1}{2}2^{[3/2]}$, respectively, to the spectrum. We have to count these maximal orbits to obtain the following theorem.

Theorem 5.7. *Let* $M = G_i \backslash \mathbb{R}^3$ *be a three-dimensional Bieberbach manifold as in Theorem 2.8. Let M carry the spin structure given by* $\delta_1, \delta_2, \delta_3 \in \{\pm 1\}$. *Then the symmetric component* $\mathcal{A}$ *of the Dirac spectrum is*

$$\mathcal{A} = \left\{\lambda_{klm}^{\pm} | (k, l, m) \in I\right\},$$

where $\lambda_{klm}^{\pm} \in \mathbb{R}$ *and* $I \subset \mathbb{Z}^3$ *are to be chosen as follows*:

G2.

(a) $\delta_1 \in \{\pm 1\}, \quad \delta_2 = 1, \quad \delta_3 = 1:$

$I = \{(k, l, m) | k, l, m \in \mathbb{Z}, m \geq 1\} \cup \{(k, l, m) | k, l \in \mathbb{Z}, l \geq 1, m = 0\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k + \frac{1}{2}\right)^2 + \frac{1}{L^2} l^2 + \frac{1}{S^2}\left(m - \frac{T}{L} l\right)^2}.$$

(b) $\delta_1 \in \{\pm 1\}, \quad \delta_2 = -1, \quad \delta_3 = 1: \quad I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 0\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k + \frac{1}{2}\right)^2 + \frac{1}{L^2}\left(l + \frac{1}{2}\right)^2 + \frac{1}{S^2}\left(m - \frac{T}{L}\left(l + \frac{1}{2}\right)\right)^2}.$$

(c) $\delta_1 \in \{\pm 1\}, \quad \delta_2 = 1, \quad \delta_3 = -1: \quad I = \{(k, l, m) | k, l, m \in \mathbb{Z}, m \geq 0\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k + \frac{1}{2}\right)^2 + \frac{1}{L^2} l^2 + \frac{1}{S^2}\left(\left(m + \frac{1}{2}\right) - \frac{T}{L} l\right)^2}.$$

(d) $\delta_1 \in \{\pm 1\}, \quad \delta_2 = -1, \quad \delta_3 = -1: \quad I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 0\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k + \frac{1}{2}\right)^2 + \frac{1}{L^2}\left(l + \frac{1}{2}\right)^2 + \frac{1}{S^2}\left(\left(m + \frac{1}{2}\right) - \frac{T}{L}\left(l + \frac{1}{2}\right)\right)^2}.$$

G3.

(a) $\delta_1 = 1: \quad I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 1, m = 0, \ldots, l - 1\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k + \frac{1}{2}\right)^2 + \frac{1}{L^2} l^2 + \frac{1}{3L^2}(l - 2m)^2}.$$

(b) $\delta_1 = -1: \quad I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 1, m = 0, \ldots, l - 1\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2} k^2 + \frac{1}{L^2} l^2 + \frac{1}{3L^2}(l - 2m)^2}.$$

G4.

(a) $\delta_1 \in \{\pm 1\}, \quad \delta_2 = 1: \quad I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 1, m = 0, \ldots, 2l - 1\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k + \frac{1}{2}\right)^2 + \frac{1}{L^2}\left(l^2 + (m - l)^2\right)}.$$

(b) $\delta_1 \in \{\pm 1\}, \quad \delta_2 = -1:$

$I = \{(k, l, m) | k, l, m \in \mathbb{Z}, l \geq 1, m = 0, \ldots, 2l - 2\},$

$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k + \frac{1}{2}\right)^2 + \frac{1}{L^2}\left(\left(l - \frac{1}{2}\right)^2 + \left(m - l + \frac{1}{2}\right)^2\right)}.$$

G5.

$$\delta_1 \in \{\pm 1\}: \quad I = \{(k,l,m) | k,l,m \in \mathbb{Z}, l \geq 1, m = 0, \ldots, l-1\},$$
$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k+\frac{1}{2}\right)^2 + \frac{1}{L^2}l^2 + \frac{1}{3L^2}(2l-m)^2}.$$

G6.

$$\delta_1, \delta_2, \delta_3 \in \{\pm 1\} \, with \, \delta_1 \cdot \delta_2 \cdot \delta_3 = 1: \quad I = \{(k,l,m) | k,l,m \in \mathbb{Z}, l \geq 0, k \geq 0\},$$
$$\lambda^{\pm}_{klm} = \pm 2\pi \sqrt{\frac{1}{H^2}\left(k+\frac{1}{2}\right)^2 + \frac{1}{L^2}\left(l+\frac{1}{2}\right)^2 + \frac{1}{S^2}\left(m+\frac{1}{2}\right)^2}.$$

For G3 the multiplicity for every $\lambda^{\pm}_{klm}$ is given by

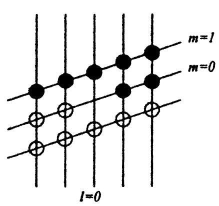
$$\mathrm{mult}(\lambda^{\pm}_{klm}, D) = 2 \cdot \#\{(k',l',m') \in I | \lambda^{\pm}_{k'l'm'} = \lambda^{\pm}_{klm}\}.$$

For all the other cases one has

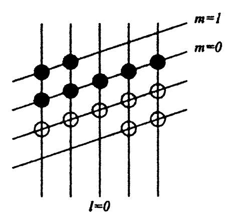
$$\mathrm{mult}(\lambda^{\pm}_{klm}, D) = \#\{(k',l',m') \in I | \lambda^{\pm}_{k'l'm'} = \lambda^{\pm}_{klm}\}.$$

Proof. We need concrete procedures to count the maximal orbits. For G2–G5 the holonomies consist of rotations around the z-axis. In these cases the orbits sit in planes which are parallel to the x-y-plane. The following pictures illustrate how to find representing elements of the orbits in these planes. They are marked by the filled circles.

G2a)



G2c)



In the case G2(a) we take the system of representatives:

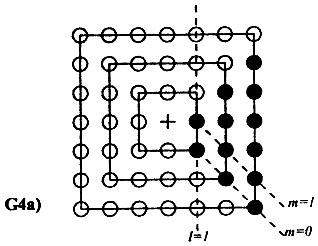
$$\{b_{klm} | (k,l,m) \in I\} \quad \text{with} \quad I \quad \text{as in the theorem},$$

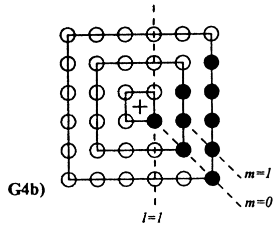
where $b_{klm} = (k + \frac{1}{2})a_1^* + la_2^* + ma_3^*$.

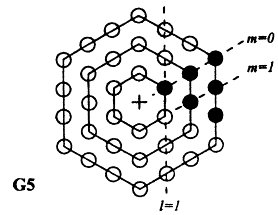
For G2(c) we choose the representatives $b_{klm} = (k + \frac{1}{2})a_1^* + la_2^* + (m + \frac{1}{2})a_3^*, k, l, m \in \mathbb{Z}, m \geq 0$.

In the cases G2(b) one has to replace l by $(l + \frac{1}{2})$ and $(m + \frac{1}{2})$ by m to get suitable b_{klm}. The case G2(d) is analogous. For these cases we choose the following representatives:

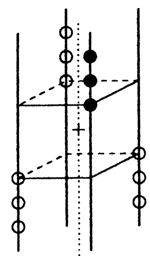
	b_{klm}	
G4(a)	$(k + \frac{1}{2})a_1^* + la_2^* + (m - l)a_3^*$	$k \in \mathbb{Z}, l \geq 1, m = 0, \ldots, 2l - 1$
G4(b)	$(k + \frac{1}{2})a_1^* + (l - \frac{1}{2})a_2^* + (m - l + \frac{1}{2})a_3^*$	$k \in \mathbb{Z}, l \geq 1, m = 0, \ldots, 2l - 2$
G5	$(k + \frac{1}{2})a_1^* + la_2^* - ma_3^*$	$k \in \mathbb{Z}, l \geq 1, m = 0, \ldots, l - 1$







For G3(a) one has the same $\Gamma^* + a_\varepsilon$ as in the case of G5. Every maximal $r(G_5)$-orbit is the disjoint union of two maximal $r(G_3)$-orbits. Therefore, we get the same spectrum as in the case of G5, but the multiplicities are doubled. For G3(b) replace $(k+1)$ by k.



Again, the case G6 differs from the other cases: every maximal orbit consists of four points which do not sit in a common plane. We take the representing elements: $b_{klm} = (k+\frac{1}{2})a_1^* + (l+\frac{1}{2})a_2^* + (m+\frac{1}{2})a_3^*$ with $m \in \mathbb{Z},\ k, l \geq 0$. □

6. Parallel spinors

The remaining section deals with parallel spinors.

Theorem 6.1. *Let M be a three-dimensional compact connected spin manifold carrying a non-zero parallel spinor. Then M is a flat torus.*

Proof. Friedrich showed in [6] that manifolds admitting non-vanishing parallel spinors are Ricci flat. In the case of dimension 3 this implies flatness. Therefore, M is Bieberbach. The kernel of the Dirac operator is non-trivial since parallel spinors are harmonic. Applying Theorem 5.1 finishes the proof. □

The last theorem gives a characterisation of flat tori in the class of Bieberbach manifolds.

Theorem 6.2. *Let $M = G\backslash\mathbb{R}^n$ be a Bieberbach manifold carrying the induced orientation and the spin structure associated to $\varepsilon : G \to \mathrm{Spin}(n)$. If the kernel of the Dirac operator has dimension $2^{[n/2]}$, M is a flat torus.*

Proof. A consequence of dimension $2^{[n/2]}$ is that $\ker(D) = \Sigma_n$. Hence for all $g \in G, \sigma \in \Sigma_n$ we have $\sigma = \varepsilon(g)\cdot\sigma$. Since the representation of $\mathrm{Spin}(n)$ on Σ_n is faithful, it follows that $\varepsilon \equiv 1$. The condition $r = \lambda \circ \varepsilon$ for spin structures implies $r \equiv 1$. This means that $G = \ker(r)$ is a lattice, and M is a torus. □

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References

[1] C. Bär, Das Spektrum von Dirac-Operatoren, Bonner Math. Schr. 217 (1991).
[2] C. Bär, The Dirac operator on space forms of positive curvature, J. Math. Soc. Jpn. 48 (1996) 69–83.
[3] L. Bieberbach, Über die Bewegungsgruppe der Euklidischen Räume 1, Math. Ann. 70 (1911) 297–336.
[4] L. Bieberbach, Über die Bewegungsgruppe der Euklidischen Räume 2, Math. Ann. 72 (1912) 400–412.
[5] L.S. Charlap, Bieberbach Groups and Flat Manifolds, Springer, New York, 1986.
[6] T. Friedrich, Zur Existenz paralleler Spinorfelder über Riemannschen Mannigfaltigkeiten, Colloq. Math. 44 (1981) 277–290.
[7] T. Friedrich, Zur Abhängigkeit des Dirac-Operators von der Spin-Struktur, Colloq. Math. 47 (1984) 57–62.
[8] T. Friedrich, Dirac-Operatoren in der Riemannschen Geometrie, Vieweg, Braunschweig, 1997.
[9] W. Hantzsche, H. Wendt, Dreidimensionale euklidische Raumformen, Math. Ann. 110 (1935) 593–611.
[10] T. Sakai, Riemannian Geometry, American Mathematical Society, Providence, RI, 1996.
[11] E.T. Whittaker, G.N. Watson, A Course in Modern Analysis, 4th Edition, Cambridge University Press, London, 1927.
[12] J.A. Wolf, Spaces of Constant Curvature, McGraw-Hill, New York, 1967.